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GOLDEN SECTION ATRIA

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Awards: Lester Ford Award, 2002.
Publications:
- Similarities in irrationality proofs for pi, ln2, zeta2 and zeta3, *The American Mathematical Monthly*, 2001.
- More true Applications of the Golden Number, in *The Golden Section, Nexus Network Journal on Architecture and Mathematics*, Volume 4, Number 1, Kim Williams Books, 2002.
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Abstract: *Several years ago, the author wrote papers about the golden section, the (in)famous number 1.618…, well aware of the far-fetched imagination in many, if not to say, most, golden section 'research'. Though it has never been shown the rectangle of width 1 and length 1.618… is the 'most elegant', he could propose a simple maximum-minimum problem of which the golden section is the solution. This problem only had a purely mathematical inspiration, until architecture student Tomas Devos recently confronted the author with a question about an architectural design with a large 'atrium' in the middle, surrounded by several auxiliary buildings. This search turned out in an unexpected option for the half of the golden section.*

1 PREVIOUS RESULT ABOUT 'OPTIMAL PROPORTIONS'

The golden number ϕ arises when a line segment of length x (>1) is divided into two pieces of lengths 1 and x-1 such that the whole length stands to 1, as 1 is to the remaining piece, x-1. Thus, the ratio x/1 must equal the ratio 1/(x-1), producing the quadratic equation $x^2-x-1 = 0$, of which $(1+\sqrt{5})/2 = 1.6180... = \phi$ is the positive solution. More generally, the positive roots of $x^2-nx-1=0$ yield the family of 'metallic means', for n=1, 2, … (de Spinadel, 1998). For n=2, it is the silver mean, $\sigma_{Ag} = 1+\sqrt{2}$, etc. This cookbook recipe yields the proportions seen in a pentagon as well as the property $\phi - 1 = \phi^{-1} = 0.618\ldots$ However, it does not explain what makes ϕ exceptional or ‘optimal’. Some pretend the metallic means would be *"the most irrational of all irrational numbers",* though they do not say when a number is more or less irrational than another number. Agreed, the representations of the metallic means as continued fraction are remarkable. Another ‘justification’ for the golden number myth is the omnipresence of the so-called ‘golden rectangle’ of width 1 and length ϕ. It would be the most elegant rectangle to most people, but there is no reliable statistical substantiation for this statement. Thus, the golden section ‘myth’ has faded because of the solid research initiated by Markovsky (1992) and Herz-Fischler (2005). Yet, it still flourishes on the internet and in art and architecture circles.

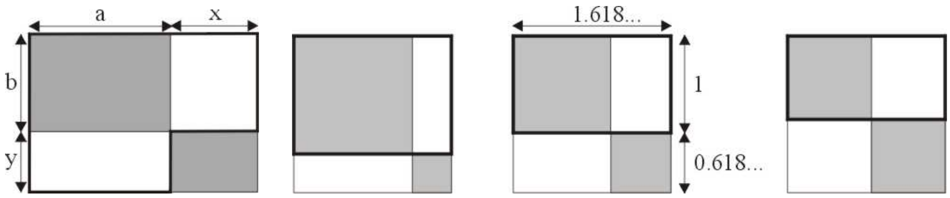

Figure 1: A rectangle is extended, yielding one which is too thick, or a perfect golden rectangle, or too thin (from left to right).

A set-up where the golden section appears as a solution to a mathematical problem was presented in previous papers (see fig. 1): consider a rectangle of arbitrary length *a* and arbitrary width *b*. Increase *a* by *x* and *b* by *y* to create a larger rectangle, of length *a*+*x* and width *b*+*y*. The new area surrounded be a tick black line is made up by three rectangles, two in white on the border of the original, and one in grey: *ab + xb + ya*. We compare it to the grey area on the main diagonal, *ab + xy*. It produces the function:

$$f(x, y) = \frac{a.b + x.b + y.a}{a.b + x.y}\,.$$

Its extreme value is computed using partial derivatives and corresponds to:

$$\left(a\frac{-1+\sqrt{5}}{2}, b\frac{-1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right) \text{ and } \left(a\frac{-1-\sqrt{5}}{2}, b\frac{-1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right).$$

They are saddle points, and the z-value is the golden number, in the first case, or the opposite of its inverse, in the second case, and this independently of the value of *a* and *b*. Actually, we can as well focus on extending a square with side 1 (= *a* = *b*) by another square of side *x* to form a larger square with side *1+ x*.

The silver section $1+\sqrt{2}$ can also be obtained through such an optimisation procedure. Starting with a square (*a*=*b*=*1*) and regarding an area at one side only, the comparison is now again made with respect to the area of the two diagonal squares. The function is $s(x)=(1+x)/(1+x^2)$, and its optimal values correspond to $-1\pm\sqrt{2}$. Here, $-1+\sqrt{2}$ is the inverse of the silver section.

The previous formulations correspond to the classical search for an ‘optimal solution’. However, the question remained if architects or artists concerned about geometric proportions would really be looking for such a ‘complicated’ maximisation.

2 A QUESTION ABOUT AN ARCHITECTURAL DESIGN

Recently, Tomas Devos, a student at the author’s Department for Architecture Sint-Lucas, confronted the author with such a question, though rather unconsciously. He intended to design a building for an architecture school, with a large ‘atrium’ or ‘plaza’ in the middle, surrounded by several auxiliary buildings. He was looking for ‘a well-thought-off proportion’ (see fig. 2).

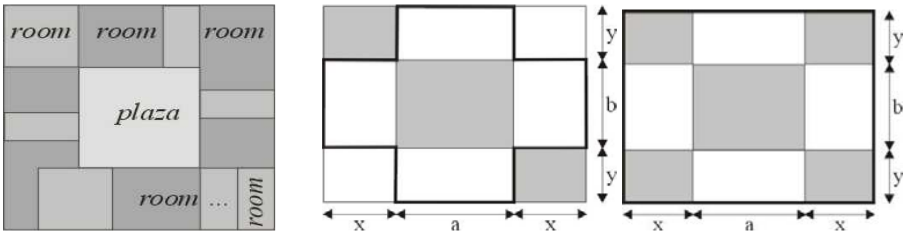

Figure 2: Student Devos’ initial sketch (left); the area surrounded by a dark black line is compared to the sum of the grey rectangles (right).

Now let us start with an atrium of length *a* and width *b*. We intend to surround it by a corridor, thus extending the rectangle with *x* along the length on both sides, and *y* along the width, again on both sides. Comparing the area surrounded by the dark line, $(a+2x)(b+2y) - 4xy$, to the area of the grey rectangles on a diagonal, ab+2xy, yields

$$f_1(x,y) = \frac{(a+2x).(b+2.y)-4xy}{a.b+2.x.y}.$$

Or else, comparing the newly created area by extending the initial rectangle on both sides by *x* and *y*, $(a+2x)(b+2y)$, to the area of all grey squares, $ab+4xy$, yields a function

$$f_2(x,y) = \frac{(a+2x).(b+2.y)}{a.b+4.x.y}.$$

In both cases an optimal solution was $f_1(a/2, b/2) = f_2(a/2, b/2) = 2$. This seemed too straightforward, and so we tried to be more creative with the above mathematics. We opted, for instance, for a combination of both above possibilities. Thus, let us compare the area surrounded by the dark line, $(a+2x)(b+2y) - 4xy$, in fig. 3, to the area of all grey squares, $ab+4xy$:

$$f_3(x,y) = \frac{(a+2x).(b+2.y)-4xy}{a.b+4.x.y}.$$

Independently of *a* and *b*, it now leads to the solution $\frac{-1+\sqrt{5}}{4}$, that is, half of the inverse of the golden section. Of course, this is the solution of the first paragraph, if the squares and rectangles are rearranged differently. However, the present set-up makes more sense from an architectural point of view – it is a natural question, as illustrated by the student’s question.

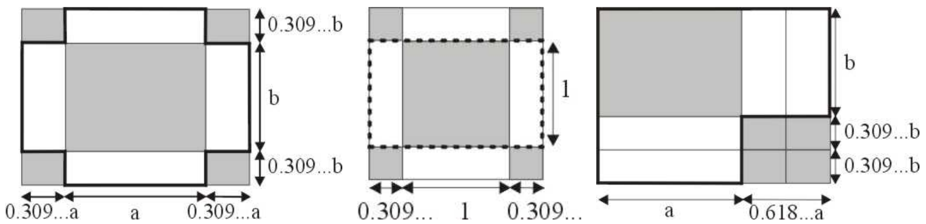

Figure 3: The solution, starting from an arbitrary rectangle (left) or from a square (middle; the dotted rectangle is a golden rectangle), and a rearrangement of the initial problem following the architecturally more sensible set-up (right).

3 EXAMPLES

Surrounding an open space by a set of buildings or rooms is an often recurring issue in architecture, and discovering a golden section relation in this matter thus is rather interesting. We do not want to go into the perilous discussion if this was done purposely or not, with or without the intention to respect the proportion. We are aware even Le Corbusier may have re-interpreted his own designs using the golden section, once he learned about it, later on.
Nevertheless, the golden section remains a famous and popular geometric proportion, so that the present suggestion of 'discovering' half of the golden section in hallways surrounding an atrium may become seminal to a new flood of (pseudo-) scientific papers about the golden section.

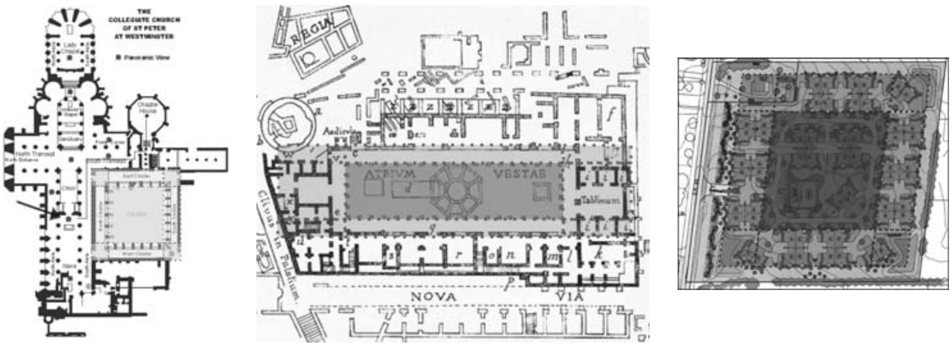

Figure 4: An example of an 'exact fit' (left), one with a more dubious interpretation (middle), and a modern example based on a parallelogram (right).

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